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Extended Abstracts

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SCM Master Thesis Day

Last October 3, with a notable attendance, we celebrated the third SCM TFM day. This is an activity organized by the Catalan Mathematical Society (SCM) which aims to facilitate those who have just graduated from a master's degree in mathematics at a Catalan university or from the common linguistic area (Xarxa Vives) to present their Final Master's Thesis. This interuniversitary activity it is about giving young master's graduates the opportunity to participate and present their first communication at a workshop, to energize the community of young mathematicians in the country that start the research, to inform about the convocation of the Galois awards and about the magazine *Reports@SCM*, and to spread the word about the mathematics master's programs of the universities of the Vives Network to students in the final year of the mathematics degree attending the day.

The day was held at the headquarters of the Institut d'Estudis Catalans and had the participation as speakers of eight students, and also with the presentation of the two master's theses awarded with the Evariste Galois 2025 prize (Pedro López and Joaquim Duran, winner and recipient), an award given by the SCM to the best final master's thesis of the previous year, in this case, 2024. In fact, one of the two winners of the 2025 Galois award presented their TFM in the 2024 edition of the SCM TFM day.

The scientific committee of the day was Enric Cosme (co-coordinator of the Master's in Mathematical Research of the UV-UPV), Simone Marchesi (editor in chief of *Reports@SCM*), Xavier Massaneda (coordinator of the Master's in Advanced Mathematics of the UB-UAB), Jordi Saludes (coordinator of the Master's in Advanced Mathematics of the UPC) and Pablo Sevilla (co-coordinator of the Master's in Mathematical Research of the UV-UPV). The organizing committee was Montserrat Alsina (president of the SCM), Josep Vives (vice-president of the SCM) and Òscar Burés and Philip Pita, current PhD students and former participants in previous editions of the day.

Reports@SCM collects in this issue the extended abstracts of the presentations of the day.

Extended Abstracts

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Densities for Hausdorff measure and rectifiability. Besicovitch's 1/2-conjecture

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Resum *(CAT)*

En aquest treball estudiem un dels conceptes centrals de la teoria geomètrica de la mesura, el de conjunt rectificable, i la seva relació amb les densitats per la mesura de Hausdorff. En aquesta interacció hi ha un dels problemes oberts més antics de la teoria: la conjectura-1/2 de Besicovitch. Estudiem una selecció de resultats rellevants, des dels articles pioners de Besicovitch [1] fins a la millora de Preiss i Tišer [7]. Després, presentem una contribució original: generalitzem a $\mathbb{R}^n$ un exemple donat originalment per Besicovitch en el pla, demostrant-ne les propietats clau i estenent així una cota inferior de la conjectura a dimensió arbitrària.

Keywords: *geometric measure theory, Hausdorff measure, rectifiability, Besicovitch's* 1/2*-conjecture.*

Abstract

One of the main concepts of geometric measure theory is that of m-rectifiable subsets of $\mathbb{R}^n$, given integers $0 < m \leq n$. They appear as a generalization of the notion of "nice" m-dimensional surfaces, such as C^1 submanifolds, or Lipschitz graphs. They are sets which, up to a set of zero $\mathcal{H}^m$-measure, are contained in a countable union of images of Lipschitz maps with domain in $\mathbb{R}^m$ (where $\mathcal{H}^m$ denotes the m-dimensional Hausdorff measure). For example, for $m = 1$, the 1-rectifiable sets are those which are contained in a countable union of rectifiable curves, again up to a set of zero $\mathcal{H}^1$-measure. On the other side of the coin, we have the purely m-unrectifiable sets, which are those that contain no m-rectifiable subset of positive $\mathcal{H}^m$-measure. One of the goals of geometric measure theory is to characterize rectifiability in terms of other geometric or analytical properties.

To that end, one of the basic tools is that of the densities for the Hausdorff measure. Consider a set $E \subset \mathbb{R}^n$ such that $0 < \mathcal{H}^s(E) < \infty$ for some $0 \leq s \leq n$, which we call an s-set. One defines the upper and lower s-densities of E at a point $x \in \mathbb{R}^n$, denoted as $\Theta^{*s}(E,x)$ and $\Theta^s_*(E,x)$ respectively, as the lim sup and lim inf as $r \to 0$ of

$$\frac{\mathcal{H}^s(E \cap B_r(x))}{(2r)^s}.$$

When both quantities coincide, the limit is called the s-density of E at x.

The densities for the Hausdorff measure and the notion of rectifiability are intimately connected. One of the most important theorems in this direction states that an m-set $E \subset \mathbb{R}^n$ is m-rectifiable if and only if the m-density of E exists and is equal to 1 at $\mathcal{H}^m$-almost all points of E. This is known as the characterization of rectifiability in terms of densities. This line of study was initiated in the pioneering work of Besicovitch [1] in 1938, where he established the result for 1-sets in the plane, i.e., the case $m = 1$ and $n = 2$. It was extended to arbitrary dimension in different stages, with the work of Moore [6], Marstrand [4] and Mattila [5].

Another point of connection between the two topics involves the lower density alone. It was proven by Besicovitch in the same article that if $\Theta^1_*(E, x) > 3/4$ for $\mathcal{H}^1$-almost all points of a 1-set E, then E is automatically 1-rectifiable. Following this idea, we define the following coefficient:

$$\sigma_m(\mathbb{R}^n) := \min\{\sigma > 0 : \text{for any } m\text{-set } E \subset \mathbb{R}^n,\ \Theta^m_*(E, x) > \sigma\ \mathcal{H}^m\text{-a.e. } x \in E \implies E \text{ is } m\text{-rectifiable}\}.$$

The previously stated result of Besicovitch translates to the bound $\sigma_1(\mathbb{R}^2) \leq 3/4$. Moreover, in the same article in 1938 he provided an example of a purely 1-unrectifiable set P which satisfies $\Theta^1_*(P, x) = 1/2$ at $\mathcal{H}^1$-almost all $x \in P$; a formal proof of this fact appeared later in a paper by Dickinson [3] in 1939. This way, they proved the lower bound $\sigma_1(\mathbb{R}^2) \geq \frac{1}{2}$. With this in mind, Besicovitch conjectured that the exact value of $\sigma_1(\mathbb{R}^2)$ is $1/2$, which is now known as *Besicovitch's* $1/2$*-conjecture*.

Further improvements to this bound have been obtained since then. In 1992, Preiss and Tišer [7] refined the estimate to $\sigma_1(\mathbb{R}^n) \leq (2 + \sqrt{46})/12 < 59/80$, which holds for all $n \geq 2$ (for all metric spaces, in fact). Recently, in 2024, Camillo De Lellis et al. [2] established that $\sigma_1(\mathbb{R}^n) \leq 7/10$, which is currently the best known upper bound.

In higher dimensions (for $m > 1$), no good upper bounds are known for $\sigma_m(\mathbb{R}^n)$. On the other hand, the same lower bound remains valid; in this work, we generalize Besicovitch's example in the plane to arbitrary dimensions, thereby showing

$$\sigma_m(\mathbb{R}^n) \geq \frac{1}{2}, \quad \text{for any } 0 < m < n.$$

This is an original contribution from this work.

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Fusion theorems and applications

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Resum *(CAT)*
En teoria de grups finits, molts resultats clàssics impliquen subgrups de Sylow. Una direcció natural és generalitzar-los mitjançant subgrups de Hall. En aquest treball, mostrem com un resultat de Wielandt permet fer-ho eficaçment. Presentem dues aplicacions: una relacionada amb el teorema de fusió d'Alperin, i una altra amb el subnormalitzador, un concepte menys conegut però amb connexions recents amb la teoria de caràcters.

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Keywords: *fusion in groups, subnormalizer.*

Abstract

In finite group theory, many results are formulated in terms of Sylow subgroups and rely heavily on the classical Sylow theorems. These results are central to the local-global philosophy of the subject, where local properties of subgroups provide valuable information about the structure of the whole group.

Whenever such theorems are established, a natural line of inquiry arises: can these results be generalized beyond Sylow subgroups? One promising direction involves replacing Sylow subgroups with Hall subgroups, which are more general but retain many desirable properties when they exist. However, such generalizations often require more sophisticated tools, since the theory of Hall subgroups is not as robust or widely applicable as Sylow theory in general finite groups.

In this work, we focus on a classical but perhaps underappreciated result by Wielandt, which proves to be a powerful instrument in extending certain Sylow-based statements to more general contexts involving Hall subgroups. Wielandt's theorem offers a unifying perspective that opens the door to new applications.

We present two main applications of this approach. The first concerns Alperin's fusion theorem, first proved by Alperin in [1], a fundamental result describing how conjugacy in a Sylow p-subgroup is controlled in terms of the local structure. This theorem is important in some conjectures in representation theory and character theory. We will show how Wielandt's result can be used to extend aspects of this theorem beyond the Sylow subgroups, providing a more flexible framework for studying fusion phenomena.

The second application involves a less well-known concept: the subnormalizer of a subgroup. This notion, mainly studied by Carlo Casolo in [2], tries to generalize the concept of normalizer. Subnormalizers offer an alternative lens through which one can examine the internal structure of a finite group. Recent developments show that this concept is not merely technical: it is connected to new conjectures in character theory and may lead to fresh insights into the interplay between subgroup structure and representation theory.

Both applications illustrate how classical tools, when viewed from a modern perspective, can be effectively repurposed to approach contemporary problems in group theory. The ideas we present highlight the ongoing relevance of results like Wielandt's theorem and demonstrate the value of re-examining classical results through new conceptual frameworks.

Acknowledgements

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Atypical values of complex polynomial functions

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Resum *(CAT)*
Des de 1983, amb el treball de Broughton, s'han introduït diverses condicions de regularitat a l'infinit per a un polinomi complex f que garanteixen l'absència de valors crítics a l'infinit, és a dir, de valors atípics de f que no són valors crítics. En aquest treball recollim les condicions de regularitat més rellevants i estudiem les relacions que hi ha entre elles. En particular, responem a dues preguntes obertes proposades per Dũng Tráng Lê i J.J. Nuño-Ballesteros a [3].

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Keywords: *complex polynomials, atypical values, critical values.*

Abstract

The topology of complex polynomial functions $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}$ has been object of considerable study in recent decades. In particular, a central goal is to understand how the topology of the fibers $f^{-1}(c)$, $c \in \mathbb{C}$, changes. In this context, the concept of *locally trivial fibrations* plays a key role. Specifically, if f is locally a trivial fibration at $c \in \mathbb{C}$, then the topology of the fibers near c remains unchanged. The points $c \in \mathbb{C}$ where f fails to be locally a trivial fibration are called atypical values of f. The set of all atypical values of f is denoted by $\operatorname{Atyp} f$. In [4], Thom proved the finiteness of the set of atypical values. However, determining precisely this set is a major open problem.

Among the atypical values, one has the critical values, *i.e.*, $f(\Sigma f) \subset \operatorname{Atyp} f$, where Σf is the set of points $x \in \mathbb{C}^n$ where $\mathrm{d}f_x = 0$. In general, this inclusion is strict. Over the past decades, several *regularity conditions at infinity* for f have been introduced in order to guarantee the equality $f(\Sigma f) = \operatorname{Atyp} f$.

The first one is the notion of *tameness*, which was introduced by Broughton in [1] and [2]. In [5], Tibăr compiles some other regularity conditions at infinity, such as the *Malgrange Condition* (which generalizes the notion of tameness) and the *ρ-regularity at infinity*, where ρ is a *control function*. The following chain of implications is well-known:

$$\begin{aligned} f \text{ is tame} &\Longrightarrow f^{-1}(c) \text{ satisfies the Malgrange Condition} \\ &\Longrightarrow f^{-1}(c) \text{ is } \rho_E\text{-regular at infinity.} \end{aligned}$$

Most recently, in [3], Dũng Tráng Lê and J.J. Nuño Ballesteros introduced the notion of *atypical values from infinity*. In this paper, they generalize the Broughton's Global Bouquet Theorem in [2]. The paper concludes by posing several open questions aimed at gaining a deeper understanding of atypical values from infinity. Namely,

1. Is it true that, if f is tame, then f does not have atypical values from infinity?
2. Does a fiber $f^{-1}(c)$ which satisfies the Malgrange Condition correspond to a value c which is not an atypical value from infinity?

In this work we review all these definitions and explain our main contribution:

$$f^{-1}(c) \text{ is } \rho\text{-regular at infinity} \Longrightarrow c \text{ is not an atypical value from infinity.}$$

Using this result, we obtain an extension of the previous chain of implications:

$$\begin{aligned} f \text{ is tame} &\Longrightarrow f^{-1}(c) \text{ satisfies the Malgrange Condition} \\ &\Longrightarrow f^{-1}(c) \text{ is } \rho_E\text{-regular at infinity} \\ &\Longrightarrow c \text{ is not an atypical value from infinity.} \end{aligned}$$

This answers the first question of the authors in [3] and gives the right implication for the second one. The other implication remains open.

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Exploring the principles of coexistence in invader-driven replicator dynamics

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Resum *(CAT)*
En aquest treball, utilitzem la "replicator equation" per explorar una de les qüestions fonamentals de la biologia evolutiva i l'ecologia: com es genera i es manté la biodiversitat? Centrant-nos en els sistemes "invader-driven", en què les interaccions o "fitnesses" de les espècies estan determinades per l'espècie invasora independentment de l'espècie envaïda, busquem relacionar les "fitnesses" amb les espècies que coexisteixen als estats finals d'equilibri. Descobrim el mecanisme que regeix la selecció d'espècies supervivents i que maximitza la resistència del sistema envers les invasions externes, i trobem que el nombre mitjà d'espècies que coexisteixen creix amb el nombre inicial d'espècies.

Keywords: *biological modelling, replicator equation, pairwise invasion fitness matrix, multi-species system, coexistence, invasion resistance.*

Abstract

Studying the non-linear and often complex dynamics of large systems of interacting species, competing or cooperating between them, can help to discover the principles that, in ecosystems, lead some species to survive and coexist, while others go extinct, to better understand of one of the central questions in ecology and evolutionary biology that remains unsolved, which is how biodiversity is generated and maintained. In the early 1970s, ecologists widely accepted that the stability and resilience observed in rich ecosystems were enhanced by complexity and biodiversity, until in 1972 the paradigm shifted completely when Robert May mathematically showed that random complexity tends to destabilise system dynamics [3]. This raised a contradiction between observation and theory known as the ecology paradox or diversity-stability debate, highlighting the need for some hidden structure or pattern in nature, such as the antisymmetric prey-predator interactions [1]. In this work, we study invader-driven interactions as a potential mechanism for the stabilization of large complex systems and we find that, under certain assumptions, invader-driven systems lead to the coexistence of species.

We use the replicator equation as a theoretical framework [4], which originated in game theory but has been widely applied to biology and epidemiology [2]. Given a system with N species, $S = \{1, 2, \dots, N\}$, consider the pairwise invasion fitness λ_i^j from species i to j, with $i, j \in S$ and $\lambda_i^i = 0$, and the invasion fitness matrix $\Lambda = (\lambda_i^j)_{i,j\in S}$. Then, the replicator equation models the time evolution of the species frequencies $\mathbf{z} = (z_1, z_2, \dots, z_N)$, with $\sum_{i\in S} z_i = 1$ and $0 \le z_i \le 1 \; \forall\, i \in S$, as

$$\dot{z}_i = \Theta z_i((\Lambda \mathbf{z})_i - \mathbf{z}^\top \Lambda \mathbf{z}) = \Theta z_i \left(\sum_{j\neq i} \lambda_i^j z_j - Q(\mathbf{z}) \right), \quad i \in S,$$

where the constant $\Theta \geq 0$ is the speed of dynamics and $Q(\mathbf{z})$ is the global mean fitness or system resistance to external invasion. In particular, we focus on invader-driven systems, in which pairwise interactions are determined by the invading species regardless of the invaded one, i.e., $\lambda_i^j = \lambda_i(1 - \delta_i^j)\ \forall\, i, j \in S$, so each species i is characterized by its active trait λ_i. We study the equilibrium states $\mathbf{z}^*$ to understand how fitnesses in the case $\lambda_i > 0\ \forall\, i \in S$ relate to the set of surviving or coexisting species at equilibrium, $S^* = \{i \in S \mid z_i^* > 0\} \subseteq S$, finding that $Q^* = Q(\mathbf{z}^*)$ plays a crucial role in the species selection process.

We prove that locally asymptotically stable equilibria are always composed of the top $n = |S^*|$ species without gaps, with fitnesses $\lambda_N \leq \cdots \leq \lambda_n \leq \cdots \leq \lambda_2 \leq \lambda_1$ and $2 \leq n \leq N$, and, furthermore, we find numerical evidence that each system contains just one of these equilibria, which in turn is a global attractor (any initial condition with all species present tends asymptotically towards it). Therefore, for each S there is a unique S^* that can be asymptotically reached by the dynamics and, hence, a unique set of species characterized by n that end up coexisting. We discover the mechanism ruling the species selection (see Figure 1), which starting with two species iteratively adds a species $i \in S$ if $Q_{i-1}^* < \lambda_i$, that is, if it can invade the previous $i - 1$, until some species n meets the condition $\lambda_{n+1} < Q_n^* < \lambda_n$. Moreover, we prove that in each step Q^* increases, $Q_{i-1}^* < Q_i^*$, so this biological process tends to maximize the system resistance to invasion.

Lastly, using this mechanism we create an algorithm that allows to find n for several invader-driven systems generated randomly with $\lambda_i \sim \mathcal{U}[0,1]$, from $N = 5$ to $N = 500$. Fitting the data we find that the mean number of coexisting species increases according to $\bar{n} = 1.381\sqrt{N}$, suggesting that invader-driven interactions could be a potential mechanism through which ecosystems stabilize and maintain biodiversity.

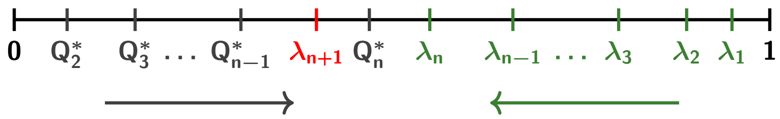

Figure 1: Species selection mechanism in invader-driven systems, $0 < \lambda_N \leq \cdots \leq \lambda_n \leq \cdots \leq \lambda_2 \leq \lambda_1 \leq 1$.

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Extended Abstracts

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Unique preduals and free objects in Banach spaces

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Resum *(CAT)*
Estudiem quan un espai de Banach té un únic predual, centrant-nos primer en les funcions holomorfes acotades al disc unitat i analitzant la demostració d'Ando. Considerem com estendre el resultat a diverses variables, on apareixen dificultats tècniques. També tractem diferents condicions suficients per garantir la unicitat i el cas de reticles de Banach.

Keywords: *unique predual, Banach space, bounded holomorphic functions, Property (X), L-embedded space, free Banach lattice.*

Abstract

A *predual* of a Banach space X is a Banach space Y such that there exists an isomorphism $Y^* \to X$. When X admits only one such space Y up to isometric isomorphism, we say that X has a *unique predual*. The problem of determining when a Banach space has a unique predual is a central one in functional analysis, starting in the works of Dixmier (1948) and Ng [5] and studied by Sakai, Ando, Godefroy, Pfitzner, and others. Classical examples of spaces with unique preduals include von Neumann algebras by Sakai (1971), the space of bounded holomorphic functions on the complex disk $H^\infty(\mathbb{D})$ by Ando [1], and separable L-embedded Banach spaces by Pfitzner [6]. Godefroy's survey [3] remains a key reference summarizing these developments and listing open problems.

This project revisits the uniqueness problem with emphasis on the space of bounded holomorphic functions, $H^\infty(U)$, defined on an open subset $U \subset \mathbb{C}^n$. We review Ando's original proof of the uniqueness of the predual of $H^\infty(\mathbb{D})$, which identifies the space $L^1(\mathbb{T})/H^1_0(\mathbb{T})$ as its unique predual. We present a detailed proof following both Ando's original formulation [1] and a later one from Garnett [2], filling several gaps left unproved in the literature.

We extend Ando's result to the case where U is a disjoint union of simply connected open subsets of the complex plane. Using Mujica's notion of the holomorphic free Banach space $G^\infty(U)$ given in [4], characterized by the universal property

$$H^\infty(U, F) \cong L(G^\infty(U), F),$$

we explicitly construct an isometric isomorphism

$$G^\infty(U) \cong \bigoplus_{\alpha \in A} G^\infty(U_\alpha),$$

where $U = \bigsqcup_{\alpha \in A} U_\alpha$. This allows us to prove that $H^\infty(U)$ has a unique predual whenever U is such a disjoint union, thus generalizing Ando's result.

We then attempt to extend the result to several complex variables, considering $H^\infty(\mathbb{T}^n)$ and $H^\infty(B_n)$. Following a different path of proving uniqueness in the one-dimensional case, we reduce the problem of proving that $H^\infty(B^n)$ has a unique predual to showing that $B_{H_0^1(\mathbb{S}_n)}$ is $\|\cdot\|_p$-closed for some $p \in (0,1)$. However, this higher-dimensional setting presents several challenges. When passing from one variable to several, the space $H_0^1(\mathbb{S}_n)$ is no longer contained in the Hardy space $H^1(\mathbb{B}_n)$, so pre-compactness arguments from Hardy space theory cannot be applied. Consequently, no conclusive results were obtained in this direction.

This work also reviews techniques guaranteeing uniqueness of preduals in broader settings. Two sufficient conditions are revisited: *Property (X)* (Godefroy–Talagrand, 1980), which ensures that X is the unique predual of X^*, and the notion of *L-embedded spaces*, for which separable cases were solved by Pfitzner [6]. Finally, we explore the Banach lattice setting, introducing the *free Banach lattice* FBL$[E]$ (Avilés–Rodríguez–Tradacete, 2015). Although we explore possible definitions for predual equivalence in this setting, we find difficulties, particularly because the space of lattice homomorphisms is not a vector space, thus leaving this problem for future research.

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Idempotent elements of the group algebra

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L'objectiu d'aquest treball és estudiar els elements idempotents centralment primitius de l'àlgebra de grup i desenvolupar un mètode per al seu càlcul en el cas de cossos finits. A partir de la teoria de representacions de grups finits i de resultats sobre mòduls, àlgebres i extensions de cossos, s'introdueix el concepte de cos d'escissió per a un grup. Finalment, s'explora com l'acció de Galois sobre l'àlgebra de grup definida sobre aquests cossos permet obtindre aquests idempotents del cos original.

Keywords: *idempotent elements, splitting fields, group algebra, Galois action, finite fields.*

Abstract

This work focuses on the study of idempotent elements of group algebras, with particular emphasis on centrally primitive idempotents. These elements are fundamental because they allow the algebra to be decomposed into simpler blocks. The importance of centrally primitive idempotents lies in the fact that each of them generates one of these blocks and, moreover, they form a basis for the centre of the algebra, which completely defines its structure.

The main objective is to develop an explicit and practical method for calculating these idempotents over fields whose characteristic does not divide the order of the group (which we will assume to be finite), and which are often not algebraically closed. This is no easy task, since many results in representation theory rely on the latter property (see [3]) and are not valid in a more general context. For this reason, we resort to the concept of a splitting field for a group (see [1, 2]), which generalises the algebraically closed field, providing a theoretical framework that guarantees the validity of many classical results, including the expression of these idempotents.

The method we develop, often known as Galois descent, consists of exploiting the expression of centrally primitive idempotents of the group algebra over a splitting field. The idea is to consider a finite Galois extension of the original field that is a splitting field for the group; in this extension, the Galois group acts on these idempotents. The expression of these idempotents is known since they are defined over a splitting field, and it can be shown that the sum of the orbits resulting from this action ultimately gives us the centrally primitive idempotents we are looking for in the original field (see [4]). This method is significantly simpler than other approaches, such as the one in [5], which relies on the computation of a division ring's dimension—a generally non-trivial task. Furthermore, we prove that both methods are equivalent by summing over a general orbit to obtain the expression given in [5].

To illustrate the procedure, we conclude with a detailed application to finite fields, where the efficiency of our approach becomes particularly evident. The practicality of the method lies not only in the simplicity of the orbit computations—thanks to the cyclic nature of the Galois group generated by the Frobenius automorphism—but also in the theoretical results previously developed in this work, which directly provide the corresponding splitting fields. In this example, we first establish the identification between characters over the splitting field and ordinary characters via Brauer characters (see again [3]), and then carry out the explicit computation of the idempotents, thereby demonstrating the applicability and strength of our self-contained approach.

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Extended Abstracts

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On nilpotency in braces and the Yang–Baxter equation

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Resum *(CAT)*
Les brides són estructures algebraiques que permeten estudiar les solucions no degenerades de l'equació de Yang–Baxter (EYB). Cada brida admet una solució no degenerada i, recíprocament, tota solució d'aquest tipus està determinada per una brida associada. Així, la classificació de les solucions no degenerades depén de l'anàlisi estructural de les brides. Les seues propietats algebraiques es corresponen amb les de les solucions, i la nilpotència permet descriure el caràcter multipermutacional d'aquestes estructures.

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Keywords: *braces, Yang–Baxter, nilpotency.*

Abstract

The Yang–Baxter equation (YBE) is a fundamental equation in theoretical physics, arising independently in the works of C. N. Yang (1967), Nobel Laureate in Physics, and R. J. Baxter, within the frameworks of quantum field theory and the study of integrable models in statistical mechanics, respectively.

The formulation of the YBE is strongly inspired by the celebrated Reidemeister moves (cf. [3]).

Consequently, the study of YBE solutions has gained significant relevance in recent decades, both because of its intrinsic importance and its applications in braid theory, braided groups, quantum groups, cryptography, and noncommutative geometry.

The multidisciplinary context of the YBE has generated great interest in the search for and classification of its solutions.

Open Problem. To find and classify the solutions of the Yang–Baxter equation.

Given the Herculean nature of this task, the Fields Medalist V. G. Drinfeld ([2]) proposed focusing on the so-called set-theoretic solutions of the YBE, a type of combinatorial solution whose geometric and symmetric character naturally gives rise to algebraic techniques.

In this work, we undertake a thorough analysis of the algebraic property of nilpotency in braces, as a clear and significant example of the translation of algebraic properties into classificatory properties of YBE solutions. We study the so-called lateral nilpotencies in braces, which have a distinct impact both on the structural analysis of braces and on the classification of solutions. In this context, a key concept of nilpotency in braces—one that has recently emerged and has a decisive impact both structurally within braces and classificatorily within YBE solutions—is central nilpotency in braces. This type of nilpotency arises with the aim of unifying both lateral nilpotencies in braces, as shown in [1], where it is demonstrated that central nilpotency in braces can be regarded as the true analogue, within brace theory, of group nilpotency.

Within group theory, the local study of nilpotency or p-nilpotency associated with a prime p has undergone substantial development following the seminal works of Hall and Higman (cf. [4]). A key concept in this context is the p-Fitting subgroup of a finite group, the largest normal p-nilpotent subgroup of the group.

The main objective and contribution of this work is the introduction and analysis of central p-nilpotency in finite braces. We conduct a comprehensive structural study of central p-nilpotency in braces, allowing us to define an appropriate p-Fitting ideal. This contribution is original within the theory and is intended to inspire further developments in this field.

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Density of hyperbolicity in families of complex rational maps

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Resum *(CAT)*
Un dels problemes oberts centrals és la densitat d'hiperbolicitat. En aquest treball ho investiguem en la dinàmica complexa unidimensional, i ens concentrem en el cas polinòmic (cas particular d'una funció racional) com a model on els mecanismes principals poden ser exposats i comprovats en detall. La via procedimental és clara: primer, la construcció de peces de puzle en un entorn del conjunt de Julia; segon, l'ús d'aquestes per definir una funció de caixa complexa; i finalment, l'aplicació de teoremes de rigidesa a aquestes. Aquest procés tradueix la informació combinatòria en rigidesa per als polinomis, demostrant que un polinomi no renormalitzable pot ser aproximat per un polinomi hiperbòlic.

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Keywords: *complex dynamics, holomorphic dynamics, rational maps, hyperbolicity, renormalisation, complex box mapping.*

Abstract

A *rational map* is a holomorphic analytic function $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ on the Riemann sphere that can be written as the quotient of two coprime polynomials; equivalently, $f(z) = \frac{P(z)}{Q(z)}$, where P, Q are complex polynomials of some degree. The degree of f is defined as $d = \max(\deg P, \deg Q)$, and we assume that $d \geq 2$. In the particular case where Q is a constant, f is just a polynomial. Rational maps of degree $d \geq 2$ form a finite-dimensional space, so exploring this parameter space is feasible. Every rational map of degree $d \geq 2$ has $2d-2$ critical points (counting multiplicity), and near these points, the map behaves like $z \mapsto z^k$, so it is highly contracting and fails to be injective. Away from the critical points, f is a local homeomorphism.

Definition (Hyperbolic rational map)**.** A rational map is said to be *hyperbolic* if all its critical points are in the basins of attracting periodic points.

Conjecture (Density of hyperbolicity)**.** *The hyperbolic rational maps form an open and dense set in the space of all rational maps of a given degree.*

Definition. We say that a map is *non-renormalisable* if it does not admit any polynomial-like restriction for any iteration with connected filled-in Julia set.

The main goal of this thesis is to deconstruct, understand all details and prove the following theorem:

Main Theorem (Theorem 1.3 in [3])**.** *Let f be a non-renormalisable polynomial of degree $d \geq 2$, without neutral periodic points. Then, f can be approximated by a sequence of hyperbolic polynomials (g_i) of the same degree.*

To tackle the problem we review and combine several fundamental tools: puzzle piece decompositions so we can consider returns and track symbolically critical orbits, Böttcher coordinates near infinity that linearize escaping behaviour, holomorphic motions to follow dynamical objects across parameters, and quasi-conformal conjugacies to transfer geometric control between maps. Also, we suppose our map is non-renormalisable: for a rational map, one demands that its critical orbits do not return in small neighbourhoods in a "periodic way". These maps are often rigid, in the sense that their combinatorial structure determines their geometry. Finally, and most importantly, we make use of complex box mappings as an induced map defined on a disjoint union of topological discs that captures return dynamics of critical orbits inside a controllable domain ("upgrade" of the famous polynomial-like maps). These are flexible enough to encode both local renormalisation behaviour and global combinatorial constraints.

The seminal paper [3] (our main reference) lacks explicit technical assumptions in its statements and proofs. This makes some of their statements, as written in that paper, not entirely correct. Some assumptions were implicit or not considered, for example, the dynamically natural property of complex box mappings. Some parts, claimed to be straightforward, are not. In [1] they clarified and fixed some results on rigidity of polynomials and box mappings, but the theorem stated above remains unclear. So for the first time in the literature of complex dynamics, we provide detailed explanations for each part of the proof of that theorem. We consider the implications and ensure validity, especially when considering the dynamically natural property of box mappings. Our aim is to review the existing literature ([4, 2]), emphasising crucial aspects, and comprehensively understand the tools required for the theorem's proof. We aim to encapsulate them in a "black-box" and use them to advance research, for instance, to establish the density of hyperbolicity in other families of rational maps. We believe this meticulous deconstruction and attention to detail can significantly contribute to the general public's understanding of the subject matter.

The proof of the Main Theorem lies on a construction of dynamically natural box mappings for non-renormalisable polynomials without neutral periodic points together with a verification of the hypotheses needed to invoke rigidity theorems. In rigid families, topologically conjugate maps are automatically more regular (e.g., quasi-conformal or conformal in complex dynamics). By another of the main theorems needed (Theorem 6.1 in [1]), combinatorially equivalent non-renormalisable dynamically natural complex box mappings are rigid, and hence, quasi-conformally conjugate. This result, along with other known or basic notions, leads to the quasi-conformal rigidity of non-renormalisable polynomials. Consequently, the original polynomial is approximated, in the uniform topology on compact sets, by hyperbolic polynomials; hence density of hyperbolicity holds for the class considered.

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